11/27.

A fibration in $s\mathcal{C}at_{\text{Bergner}}$ $\quad f : \mathcal{C} \to \mathcal{D}$

$$\begin{cases} \text{(i) } \mathcal{C}(x,y) \longrightarrow \mathcal{D}(fx, fy) & \text{is a Kan fibration.} \\ \text{(ii) } ho\mathcal{C} \longrightarrow ho\mathcal{D} & \text{is an isofibration.} \end{cases}$$

In the case $\mathcal{D} = \underline{0}$ — $ob = \{*\}$, $\mathrm{Hom}(*,*)_\bullet = \Delta^0$

then (i) $\Rightarrow$ (ii)

$ho(\underline{0}) = [0] = \{ * \circlearrowleft^{id} \}$.

$\Rightarrow$ $\begin{array}{c} ho(\mathcal{C}) \\ \downarrow \\ [0] = ho(\mathcal{D}) \end{array}$ is an iso fib. iff everything in $ho\mathcal{C}$ is a iso.

iff $ho\mathcal{C}$ is a gpd

iff $\mathcal{C}$ is a ∞-gpd $\Updownarrow$ (!)

iff $\mathcal{C}$ is a Kan cpx-en.

X Kan enriched $\Rightarrow$ $\begin{array}{c} X \\ \downarrow \\ \underline{0} \end{array}$ is a fibration in $s\mathcal{C}at_B$.

WTS (i) & (ii)
$\quad\quad$ ✓ $\quad$ ✓

(K is a Kan cpx.)

$\Updownarrow$

$\left(\begin{array}{c} K \\ \downarrow \\ \Delta^0 \end{array} \text{ is a Kan fibration} \right)$

$$\begin{tikzcd} s\mathcal{C}at \arrow[r, "N^{hc}"] \arrow[d, "ho"'] \arrow[dr, phantom, "\cong"] & sSet \arrow[dl, "h"] \\ \mathcal{C}at & \end{tikzcd}$$

$ho N^{hc} \overset{?}{=} ho$

$$\begin{tikzcd} s\mathcal{C}at \arrow[d, shift right, "N"'] \arrow[r, "ho"] & \mathcal{C}at \\ sSet \arrow[u, shift right, "\mathfrak{C}"'] \arrow[ur, "h"'] & \end{tikzcd}$$

$\mathcal{C} \longmapsto N^{hc}\mathcal{C}$ — $0 = ob\,\mathcal{C}$; $1 = \mathrm{Hom}_{sSet}(\Delta^1, N\mathcal{C}) = \mathrm{Hom}_{s\mathcal{C}at}(\mathfrak{C}[1], \mathcal{C})$; $2 \; \ldots$

$\mathcal{C} \downarrow ho\mathcal{C}$ — $ob = ob\,\mathcal{C}$; $\mathrm{Hom}(x,y) = \pi_0 \mathcal{C}(x,y)_\bullet$

$N^{hc}\mathcal{C} \downarrow hN^{hc}\mathcal{C}$ — $ob = N^{hc}\mathcal{C}_0 = ob\,\mathcal{C}$; $\mathrm{Hom}_{hN^{hc}\mathcal{C}}(x,y) = N^{hc}\mathcal{C}_1 / \sim_2$

When $\mathcal{C}$ is Kan enriched, there's mapping spaces. $\mathrm{Map}_{\mathcal{C}}(x,y)$

& $h\mathcal{C}$ has Hom sets given by connected components. $\mathrm{Hom}_{h\mathcal{C}}(x,y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(x,y)$.

$\mathrm{Hom}_{sSet}(\mathfrak{C}[1], \mathcal{C}) / \sim \overset{?}{=} \pi_0 \mathrm{Map}_{\mathcal{C}}(x,y)$.

$(0 \to 1)$

$\mathfrak{C}[1] =$

sCat	sSet.
$\mathfrak{C}[n]$	Δ^n
$\mathfrak{C}[1]$	Δ^1

$$\begin{tikzcd} \mathcal{C} \arrow[r, mapsto, "N\mathcal{C}"] & {} \\ sCat \arrow[rr] \arrow[dr, "ho"'] & & sSet \arrow[dl, "h"] \\ & Cat & \end{tikzcd}$$

Given an ∞-cat $\mathcal{C}$.., $x,y \in ob(\mathcal{C})$.

$\mathrm{Hom}_{h\mathcal{C}}(x,y) \cong \pi_0 \mathrm{Map}_{N\mathcal{C}}(x,y) = \pi_0 \mathrm{Hom}_{\mathcal{C}}(x,y)$

$\mathrm{Map}_{N\mathcal{C}}(x,y) \overset{?}{=} \mathrm{Hom}_{\mathcal{C}}(x,y)$

Mapping space of a quasi-cat.

$$\begin{tikzcd} \mathrm{Map}_{N\mathcal{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[d, "(s,t)"] \\ \Delta^0 \arrow[r, "(x,y)"] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

a p.b. of ssets.

$\mathcal{C} \in sCat$

$\mathrm{Hom}_{\mathcal{C}}(x,y) \in sset.$

$$\begin{tikzcd} sCat \arrow[r, shift left] & sSet \arrow[l, shift left] \\ KanCat \arrow[u, hook] \arrow[r, shift left, "N"] \arrow[dr, bend right, "ho"'] & qCat \arrow[u, hook] \arrow[l, shift left, "\mathfrak{C}"] \arrow[dl, bend left, "h"] \\ & Cat \end{tikzcd}$$

$$\begin{tikzcd} KanCat \arrow[dr, "ho"'] & qCat \arrow[l, "\mathfrak{C}"'] \arrow[dl, "h"] \arrow[d, phantom, "\cong"] \\ & Cat \end{tikzcd}$$

thesis 2.4.1.15 ?

$$\begin{tikzcd} \mathcal{C} \arrow[r, mapsto, "N^{hc}"] & N\mathcal{C} \arrow[r, mapsto] & \mathfrak{C}N\mathcal{C} \simeq \mathcal{C} \\ KanCat \arrow[r] \arrow[dr, bend right] & qCat \arrow[r, "\mathfrak{C}"] \arrow[d] & KanCat \arrow[dl, bend left] \\ & Cat & \end{tikzcd}$$

$\Rightarrow hN^{hc}\mathcal{C} \cong ho^{sCat}(\mathfrak{C}N\mathcal{C})$

$\cong ho^{sCat}(\mathcal{C})$

from Cisinski/thesis.

since $\mathfrak{C}N\mathcal{C} \simeq \mathcal{C}$ Quillen adjunction. $\mathfrak{C} \dashv N$

§1. Homotopy theories, model cats, ∞-cats.

A htpy theory is "(C, W)". [We can describe htpy theories as a htpy thy.]

two ways to "model them"

1.1. ① model cats.

1.2. ② ∞-cats.

1.2.1 ⓐ kan-cat $\cap$ scat.

1.2.2 ⓑ qcats $\cap$ ssets.

RelCat $\longrightarrow$ scat $\longrightarrow$ qcat,

$$C \underset{Q}{\sim} D \longmapsto L^H C \underset{DK}{\sim} L^H D \longmapsto N L^H C \underset{w.eq.}{\sim} N L^H D$$

by $\mathrm{sCat} \underset{Q}{\simeq} \mathrm{sSet}$ (Bergner, Joyal) ?? [HTT, 2.2.5.1].